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DECOMPOSING NORMED UNITS IN COMMUTATIVE MODULAR GROUP RINGS

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Abstract. We find a necessary and sufficient condition only in terms of the abelian p-group G and the perfect field F of characteristic p for the group $V(FG)$ of normalized units in a group ring FG to be decomposed as $G(1+I^2(FG;G))$, where $I(FG;G)$ is the augmentation ideal in FG. This sheds some light on the long-standing *Direct Factor Problem*.

1. Introduction

Everywhere in the text, suppose F is a field of nonzero characteristic p and G is an abelian group written multiplicatively as it is the custom when exploring group rings. So, FG denotes the group ring of G over F with normalized unit group $V(FG)$ and nil-radical $N(RG)$. For any ideal I of FG set $I^n = \underbrace{I \dots I}_{n}$, where $n \in \mathbb{N}$, and $I^\omega = \cap_{n<\omega} I^n$. As usual, C_p will denote the cyclic group of order and cardinality p. Our epimorphisms mean surjective homomorphisms. All other undefined and unstated explicitly notions and notations follow essentially those from [6] and [7].

The leitmotif of this paper is to find a criterion only in terms of F and G when the equality $V(FG) = G(1+I(FG;G).I(FG;G))$ holds, provided G is a p-group. Iterating, it will be very useful for applications to the *Direct Factor Problem* for modular group rings to know when

$$V(FG) = G(1 + I(FG;G).I(FG;G).\cdots .I(FG;G))$$

is fulfilled, where the number of the ideal $I(RG;G)$ may vary.

2. Main Result

We begin here with the following simple but useful technicality.

Lemma 2.1. *Let R be a commutative ring with identity of prime characteristic p, G an abelian group and A an abelian p-group. If the map $G \to A$ is an epimorphism, then its element-wise extending map $V(RG) \to V(RA)$ is also an epimorphism.*

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Proof. Since $G \to A$ is a homomorphism, it is easily checked that $V(RG) \to V(RA)$ is also a homomorphism with kernel $1 + I(RG; A)$. But $I(RG; A)$ is a nil-ideal and thus $1 + I(RG; A)$ is a p-group. Now the desired epimorphism can be readily established. □

Recall that a field F is said to be *perfect* if $F = F^p$.

Theorem 2.2. *Let F be a perfect field of characteristic $p > 0$ and let G be an abelian p-group. Then the equality*

$$V(FG) = G(1 + I^2(FG; G))$$

holds if, and only if, the following two conditions are true:
(1) $G = G^p$;
(2) $G \neq G^p$ and $F = \mathbb{Z}_p$.

Proof. "**Necessity**". Given $G \neq G^p$, we observe that there is a sequence of two epimorphisms $G \to G/G^p \to C_p$, where the first one is the canonical homomorphism whereas the second one is the canonical projection, and so, resultantly, there is an epimorphism $G \to C_p$. Utilizing Lemma 2.1, it can be extended to the epimorphism $V(FG) \to V(FC_p)$ which sends $I(FG; G)$ to $I(FC_p; C_p)$, because $V(FG) = 1 + I(FG; G)$ and $V(FC_p) = 1 + I(FC_p; C_p)$. Consequently, the ratio $V(FG) = G(1 + I^2(FG; G))$ implies the ratio $V(FC_p) = C_p(1 + I^2(FC_p; C_p))$. Clearly, $I(FC_p; C_p)$ is a linear space over F as well as F is a one-dimensional linear space over itself. We will now construct a linear map $\Phi : I(FC_p; C_p) \to F$ such that Φ sends $I^2(FC_p; C_p)$ to $\{0\}$. To that goal, set $C_p = \langle a \rangle$ with $a^p = 1$. Define

$$\Phi(\sum_{1\leq i\leq p} f_i(a^i - 1)) = \sum_{1\leq i\leq p} if_i,$$

where $f_i \in F$. It is self evident that this is a correctly defined map between linear spaces, because for any $d \in C_p$ we have $f_id(a^i - 1) = f_i(da^i - 1) - f_i(d - 1)$ and since $d = a^l$ for some positive integer l with $1 \leq l \leq p$ it must be that $f_id(a^i - 1) = f_i(a^{i+l} - 1) - f_i(a^l - 1)$, and we are set. Moreover, because of the formula, $(b - 1)(c - 1) = (bc - 1) - (b - 1) - (c - 1)$ for some $b, c \in C_p$, say $b = a^j$ and $c = a^k$ with $j, k \in [1, p]$, it follows that $\Phi((b - 1)(c - 1)) = \Phi((a^{j+k}-1)-(a^j-1)-(a^k-1)) = (j+k)\cdot 1-j\cdot 1-k\cdot 1 = j\cdot 1+k\cdot 1-j\cdot 1-k\cdot 1 = 0$, where 1 is the identity element of F. Therefore, since products of the type $(b-1)(c-1)$ form a basis for $I^2(FC_p; C_p)$, one deduces that $\Phi(I^2(FC_p; C_p)) = \{0\}$, as wanted.

Furthermore, for any $f \in F$, we consider the normalized unit $1+f(a-1)$ which can be written like this:

$$1 + f(a - 1) = b(1 + z),$$

where $b \in C_p$ and $z \in I^2(FC_p; C_p)$.

Thus $f(a-1) = (b-1) + bz$ with $b = a^j$ for some $1 \leq j \leq p$. Since bz lies in $I^2(FC_p; C_p)$, acting by Φ on the both sides of this equality, we derive that $f = j \cdot 1$, where $1 \in F$. Hence $F \cong \mathbb{Z}_p$, and we are finished.

"**Sufficiency**". One observes that $V(FG) = 1 + I(FG; G)$. Firstly, if G is divisible, then for each $g \in G$ we have that $g = h^p$ for some $h \in G$, so that $1 - g = 1 - h^p = (1-h)^p \in I^2(FG; G)$, because $p \geq 2$. Since the elements $1 - g$ form a basis for $I(FG; G)$, we deduce that $I(FG; G) = I^2(FG; G)$ and we are done.

Secondly, assume that G is not p-divisible and F is the simple field of p elements. Given an arbitrary element $x \in V(FG)$, without loss of generality, one can write in view of the formula $V(FG) = 1 + I(FG; G)$ that

$$x = 1 + k_1 g_1 (a_1 - 1) + \cdots + k_s g_s (a_s - 1),$$

where $1 \leq k_1, \cdots, k_s \leq p-1$; $g_1, a_1, \cdots, g_s, a_s \in G$; $s \in \mathbb{N}$.

Since $k_i g_i (a_i - 1) = k_i (g_i - 1)(a_i - 1) + k_i (a_i - 1)$ for all $i \in [1, s]$, by repeating the summands we may with no harm of generality assume that $k_1 = \cdots = k_s = 1$ and thus we need to consider only the element

$$y = 1 + (a_1 - 1) + \cdots + (a_s - 1).$$

Furthermore, because of the reduction formula $(a_i - 1) + (a_j - 1) = (a_i - 1)(1 - a_j) + (a_i a_j - 1)$ which decreases the summands of the basis type $w - 1$ for some $w \in G$, we may assume by induction that $s = 2$. Therefore, $y = 1 + (a_1 - 1) + (a_2 - 1) = a_1 a_2 + (a_1 - 1)(1 - a_2) = a_1 a_2 (1 + a_1^{-1} a_2^{-1} (a_1 - 1)(1 - a_2)) \in G(1 + I^2(FG; G))$, as required.

□

The next immediate consequence is somewhat surprising.

Corollary 2.3. *If G is an abelian p-group, then the following equality is valid:*

$$V(\mathbb{Z}_p G) = G(1 + I^2(\mathbb{Z}_p G; G)).$$

We are now in a position to prove the following:

Proposition 2.4. *Suppose that F is a perfect field of characteristic $p \neq 0$ and G is an abelian p-group. Then the equality*

$$V(FG) = G(1 + N(FG).I(FG; G))$$

is true if, and only if, the following two items hold:

(1) $G = G^p$;

(2) $G \neq G^p$ and $F = \mathbb{Z}_p$.

Proof. Appealing to [5], one infers that $N(FG) = I(FG; G)$ and thus, we henceforth can apply Theorem 2.2 to conclude the claim. $\square$

Remark. In [1] it was obtained a result concerning the decomposition $V(RG) = G \times (1 + N(R)G.I(RG; G))$, where R is a commutative ring with identity of prime characteristic p and G is an abelian group. In [2] it was established a slight generalization of the preceding result. Note that the inclusion $N(R)G \subseteq N(RG)$ is always fulfilled, so that Proposition 2.4 could be considered as an eventual expansion of the presented results in [1] and [2].

In [3] it was proved a result about the validity of the decomposition $V(RG) = GV(RG_0)(1 + N(RG).I(RG; G))$, where R is an arbitrary commutative ring and G is an arbitrary abelian group with maximal torsion subgroup G_0.

In [4] it was shown that the general validity of the formula $V(RG) = GV(RG_0)(1+ N(R)G.I(RG; G))$ depends only on some restrictions on the commutative ring R and the abelian group G.

3. Problems

We finish off with some questions.

Problem 1. Suppose R is a commutative ring with identity and positive characteristic (in particular, of prime characteristic p) and suppose G is an abelian group (in particular, a p-group). Find a necessary and sufficient condition to hold the relationship $V(RG) = G(1 + I^\omega(RG; G))$.

Concerning the classical *Direct Factor Problem* in the theory of modular group rings, namely that G separates as a direct factor of $V(RG)$ whenever R is a commutative ring with identity of prime $char(R) = p$ and G is an abelian p-group, one can ask when the intersection $G \cap (1 + I^\omega(RG; G)) = \{1\}$ holds. We should provide that G is a separable p-group, that is, $G^{p^\omega} = \cap_{n<\omega} G^{p^n} = \{1\}$, because otherwise (e.g., if $G \neq \{1\}$ is divisible), then once have that $1 \neq g = a_n^{p^n} = 1 + (a_n^{p^n} - 1) = 1 + (a_n - 1)^{p^n}$ for each $n \in \mathbb{N}$ and some $a_n \in G$, whence $g \in G \cap (1 + I^\omega(RG; G))$ because $p^n > n$, as expected.

In that aspect, we can mention some positive facts about the possible truth of $G \cap (1 + I^2(RG; G)) = \{1\}$.

• It is well known that if $R = \mathbb{Z}$, then the mapping $G \to I(RG; G)/I^2(RG; G)$ given by $g \to g - 1 + I^2(RG; G)$ is an isomorphism, and thus the above intersection is true in this case.

• Now suppose that R has characteristic 0, but has an element r such that $2r^2 = 1$, and suppose that $1 \neq g \in G$ with $g^2 = 1$. Then both $r(g - 1)$ and $r(1 - g)$ are in $I(RG; G)$, but one may write that $g = 1 + r(g - 1).r(1 - g)$, thus it is false in this case.

• Let p be a prime and suppose that R has characteristic p. Suppose also that $g \in G$ such that $g^p \neq 1$. Note that $g-1$ and $(g-1)^{p-1}$ are both in $I(RG;G)$, but $g^p = 1 + (g-1).(g-1)^{p-1}$, thus again false.

• If R is the field with two elements and G has order 2 with generator g, then $I(RG;G) = \{0, g-1\}$ and hence $I^2(RG;G) = \{0\}$, thus the statement about the intersection is true in this case.

More generally, the statement could be verified as it is true or not for R being a field of $char(R) = 2$ and $G = \langle 1, g \ : \ g^2 = 1 \rangle$.

Problem 2. Let R be a commutative ring with identity and positive characteristic (in particular, of prime characteristic p) and let G be an abelian group (in particular, a p-group). Find a criterion for the truthfulness of the relation $V(RG) = G(1 + N(RG).I(RG;G))$.

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